SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

DECEMBER 2004

HSC ASSESSMENT TASK #1

Mathematics

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used
- All necessary working should be shown in every question

Total Marks – 62

- All Questions may be attempted

Examiner – *A. M. Gainford*

Question 1. (15 Marks) (**Start a new booklet**.)

Marks

(a) Express as a common fraction in lowest terms: **1**

$$\left(\frac{6^{10}\times 35^{8}}{14^{8}\times 15^{10}}\right)^{\frac{1}{2}}$$

(b) Calculate the probability of obtaining a total of 10 or more when two standard dice are rolled. **1**

(c) Factorise completely $6x^2 - 13x - 5$. **1**

(d) Sketch on the number plane the graph of the function $y = \log_2 x$ in the domain $0 < x \leq 8$. **1**

(e) Write an equation for the parabola with vertex (0, 0) and directrix $y = -3$. **1**

(f) Solve for *x*: $2\log_5 3 = \log_5 x - \log_5 6$. **1**

(g) Find the 108th term of the arithmetic series $-8 - 4\frac{1}{2} - 1 + 2\frac{1}{2} + \ldots$ **1**

(h) Find $\log_7 128$, correct to 4 decimal places. **1**

(i) Find $\lim_{x\to 0}\frac{2x^2 - x}{x}$. **2**

(j) Evaluate $\sum_{r=1}^{4} 2^{-r}$ **1**

(k) For which values of *k* does $x^2 - kx + 4$ have no real zero. **2**

(l) Given the expression $3x^2 + 12x + 5$: **2**

(i) Find the value of *x* for which the expression has its minimum value.

(ii) State the minimum value of this expression

Question 2. (15 Marks) (**Start a new booklet**.)

Marks

(a) Find and simplify the derivative of each of the following: **6**

(i) $x^3 - 4x^2 - 3$

(ii) $\left(x^2 - 1\right)^{10}$

(iii) $x\sqrt{x-1}$

(iv) $\dfrac{2x-1}{x+1}$

(b) Consider the parabola with equation $y = \frac{1}{4}\left(x^2 + 2x + 13\right)$ **4**

(i) Write the equation in the form $(x-h)^2 = 4a(y-k)$.

(ii) State the vertex and focus.

(iii) Sketch the curve.

(c) Express the arithmetic series $1+3+5+7+\ldots+199$ in sigma notation **1**

(d) A fair tetrahedral die has its faces marked 0, 1, 2, and 3, and the bottom face is noted. An experiment consists of rolling two such dice, and multiplying the outcomes. **4**

(i) Use a table to enumerate the sample space of this experiment.

(ii) State the probability that the result is 0.

(ii) State the probability that the result is greater than 4.

Question 3. (15 Marks) (**Start a new booklet**.)

Marks

(a) Find the values of *A*, *B* and *C* if $x^2-4x+3 \equiv Ax(x-1)+Bx+C$. **3**

(b) An urn contains 6 black balls and 4 white balls. Two are drawn without replacement. **2**
Find the probability that

(i) both balls are black

(ii) at least one is black.

(c) If $\log_a x = p$, $\log_a y = q$, and $\log_a z = r$, find an expression in terms of *p*, *q* and *r* for: **2**

$$3\log_a\left(\frac{x^2y}{\sqrt{z}}\right)$$

(d) Show that the equation $(x-p)(x-q)=r^2$ always has real solutions for real *p*, *q* and *r*. **2**

(e) The fifth term of a geometric series is 2, and the eighth term is $\frac{2}{27}$. **3**
Find

(i) the first term

(ii) the common ratio

(iii) the sum to infinity.

(f) The quadratic equation $x^2-5x+3-3a=0$ has roots α and β. **3**

(i) Find $\alpha+\beta$, and $\alpha\beta$ in terms of *a*.

(ii) Given that the roots differ by 11, find the value of *a*.

Question 4. (15 Marks) (**Start a new booklet**.)

Marks

(a) Find the co-ordinates of the point on the curve $y = x^3 + 6x^2 + 12x + 3$ where the gradient of the tangent is zero. **2**

(b) Solve the quadratic inequality $x^2 + x - 2 \leq 0$. **2**

(c) The positive multiples of 7 are 7, 14, 21, …. **2**

(i) What is the largest multiple of 7 less than 1000?

(ii) What is the sum of the positive multiples of 7 less than 1000?

(d) Solve the equation $x^4 - 5x^2 + 4 = 0$. **2**

(e) Consider the curve with equation $y = x^2 + x$: **3**

(i) Find the gradient of the tangent at the point on the curve where $x = 1$.

(ii) Write in general form the equations of the tangent and normal to the curve at this point.

(f) Ilya invests \$50 000 in an account which earns 8% interest, compounded annually. He intends to withdraw \$*M* at the end of each year, immediately after the interest has been paid. He wishes to be able to do this for exactly 20 years, so that the account will then be empty. **6**

(i) How much money does he have in the account immediately after he has made his first withdrawal?

(ii) Write an expression in terms of M for the amount of money in the account, immediately after his 20th withdrawal.

(iii) Calculate the value of M which leaves his account empty after the 20th withdrawal.

(iv) Suppose Ilya wished to be able to withdraw \$8000 per year for the 20 years. By using your calculator alone, estimate, to the nearest percent, the interest rate he would then need to earn on his account.

End of paper.

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

December 2004

Year 11

HSC Assessment Task #1

Mathematics

Sample Solutions

Question	**Marker**
1	Mr Fuller
2	Mr Boros/Ms Roessler
3	Mr Dowdell
4	Ms Opferkuch

Question 1

(a) $\left(\dfrac{6^{10}\times 35^{8}}{14^{8}\times 15^{10}}\right)^{\frac{1}{2}} = \left(\dfrac{3^{10}\times 2^{10}\times 7^{8}\times 5^{8}}{2^{8}\times 7^{8}\times 3^{10}\times 5^{10}}\right)^{\frac{1}{2}}$

$= \left(\dfrac{2^{2}}{5^{2}}\right)^{\frac{1}{2}}$

$= \dfrac{2}{5}$

(b) P = P(46) + P(55) + P(56) + P(64) + P(65) + P(66)

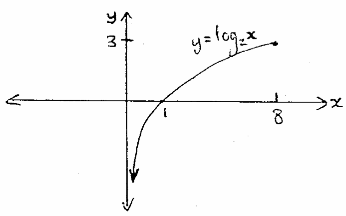
$= \dfrac{1}{36}\times 6$

$= \dfrac{1}{6}$

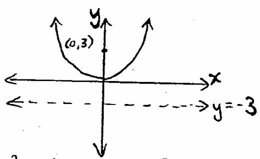
(c) $6x^2 - 13x - 5 = \dfrac{(6x-15)(6x+2)}{6}$ × -30, + -13, -15, +2

$= (2x-5)(3x+1)$

(d)



(e)



$x^2 = 4ay \qquad a = 3$

$x^2 = 12y$

(f) $2\log_5 3 = \log_5 x - \log_5 6$

$\log_5 3^2 = \log_5\left(\dfrac{x}{6}\right)$

$\therefore 9 = \dfrac{x}{6}$

$x = 54$

(g) $T_n = a + (n-1)d$

$T_{108} = -8 + (108-1)\times\left(\dfrac{7}{2}\right)$

$= 366\frac{1}{2}$

(h) $\log_7 128 = \dfrac{\log_{10} 128}{\log_{10} 7}$

$= 2\cdot 4935$

(i) $\lim\limits_{x\to 0}\dfrac{2x^2 - x}{x} = \lim\limits_{x\to 0}\dfrac{x(2x-1)}{x}$

$= \lim\limits_{x\to 0} 2x - 1$

$= 2(0)-1$

$= -1$

(j) $\sum\limits_{r=1}^{4} 2^{-r} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$

$= \dfrac{15}{16}$ or $0\cdot 9375$

(k) $\Delta = b^2 - 4ac$

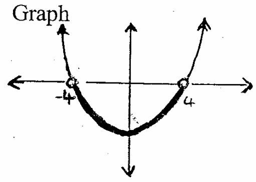
$= (-k)^2 - 4(1)(4)$

$= k^2 - 16$

No real zero when $\Delta < 0$

$k^2 - 16 < 0$

Graph



$-4 < k < 4$

(l) (i) axis of symmetry

$x = \dfrac{-b}{2a}$

$= \dfrac{-12}{2(3)}$

$= -2$

Value of x for which the expression has its minimum value is -2

(ii) sub $x = -2$ into $3x^2 + 12x + 5$

$3(-2)^2 + 12(-2) + 5$

$= -7$

Minimum value is -7

Q2 (a)

(i) $3x^2 - 8x$

(ii) $10(x^2-1)^9 . 2x = 20x(x^2-1)^9$

(iii) $y = x(x-1)^{\frac{1}{2}}$

$y' = uv' + vu'$

$= \dfrac{x}{2\sqrt{x-1}} + \sqrt{x-1}$

Simplifying

$y' = \dfrac{x}{2\sqrt{x-1}} + \dfrac{2(\sqrt{x-1})(\sqrt{x-1})}{2\sqrt{x-1}}$

$= \dfrac{x}{2\sqrt{x-1}} + \dfrac{2(x-1)}{2\sqrt{x-1}}$

$= \dfrac{3x-2}{2\sqrt{x-1}}$

(iv) $y' = \dfrac{vu' - uv'}{v^2}$

$= \dfrac{(x+1)\times 2 - (2x-1)\times 1}{(x+1)^2}$

$= \dfrac{2x+2-2x+1}{(x+1)^2}$

$= \dfrac{3}{(x+1)^2}$

(b) (i) $y = \frac{1}{4}(x^2+2x+13)$

$4y = x^2+2x+13$

$4y = x^2+2x+1 +13-1$

$4y = (x+1)^2+12$

$4y-12 = (x+1)^2$ ①

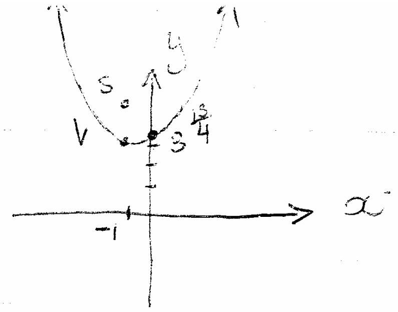
So $(x+1)^2 = 4(y-3)$ //

(ii) $V(-1, 3)$ // ①

since $a = 1$ ①

$S(-1, 4)$ //

So (b) (iii)

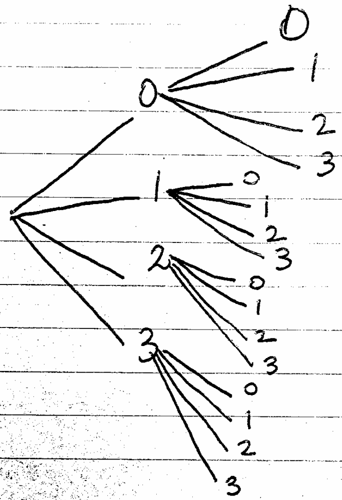


(1)

(c) $1+3+5+7+ \cdots +199 = \sum_{t=1}^{100} (2t-1)$ (1)

(d)

(i)



	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

// (2)

(ii) $\frac{7}{16}$ // (1)   (iii) $\frac{3}{16}$ / (1)

3(a) $x^2+4x-3 \equiv Ax(x-1)+Bx+C$

Let $x=0$: $-3 = C$

$\therefore x^2+4x-3 = Ax(x-1)+Bx-3$

Let $x=1$

$\therefore 1+4-3 = B-3$

$\therefore B = 5$

$\therefore x^2+4x-3 \equiv Ax(x-1)+5x-3$

Equating leading co-efficients

$A = 1$

$\therefore A=1, \quad B=5, \quad C=-3$ (2)

(b) (i) $P(BB) = \frac{6}{10} \times \frac{5}{9}$

$= \frac{1}{3}$ 1

(ii) P(at least one black)

$= 1 - P(WW)$

$= 1 - \frac{4}{10} \times \frac{3}{9}$

$= \frac{13}{15}$ 2 (3)

(c) $3 \log \left(\frac{x^2 y}{\sqrt[2]{3}}\right)$ (assuming base a)

$= 3(\log x^2 + \log y - \log \sqrt{z})$

$= 3(2 \log x + \log y - \frac{1}{2} \log z)$

$= 3(2p + q - \frac{1}{2} r)$

$= 6p + 3q - \frac{3}{2} r$ (2)

(d) $(x-p)(x-q) = r^2$

$\therefore x^2-(p+q)x+pq = r^2$

$\therefore x^2-(p+q)x+pq-r^2 = 0$

For real solutions, $\Delta \geq 0$

$\Delta = (p+q)^2 - 4 \times 1(pq-r^2)$

$= p^2+2pq+q^2-4pq+4r^2$

$= p^2-2pq+q^2+4r^2$

$= (p-q)^2+4r^2$ (2)

≥ 0 for all real p, q, r

(sum of 2 non-negative nos)

(e) $T_5 = ar^4 = 2$

$T_8 = ar^7 = \frac{2}{27}$

$\therefore r^3 = \frac{2}{27} \div 2$

$= \frac{1}{27}$

$\therefore r = \frac{1}{3}$ (ii) 1

$\therefore a \times (\frac{1}{3})^4 = 2$

$\therefore \frac{a}{81} = 2$

$\therefore a = 162$ (i) 1

(iii) $S = \frac{a}{1-r}$

$= \frac{162}{1-\frac{1}{3}}$

$= 243$ 1 (3)

(f) (i) $\alpha+\beta = 5$

$\alpha\beta = 3-3a$ 1

(ii) $\alpha-\beta = 11$ (letting α be larger root)

$\therefore (\alpha-\beta)^2 = 121$

$\therefore \alpha^2-2\alpha\beta+\beta^2 = 121$

$\therefore \alpha^2+2\alpha\beta+\beta^2 = 121+4\alpha\beta$

$\therefore (\alpha+\beta)^2 = 121+4(3-3a)$

$\therefore 25 = 121+4(3-3a)$

$\therefore -96 = 12(1-a)$

$\therefore -8 = 1-a$

$\therefore a = 9$ 2 (3)

15

4 (a) $y = x^3 + 6x^2 + 12x + 3$

$m_{tan} = \frac{dy}{dx} = 3x^2 + 12x + 12 = 0$

$x^2 + 4x + 4 = 0$

$(x+2) = 0$

$x = -2$

$\therefore y = (-2)^3 + 6(-2)^2 + 12(-2) + 3$

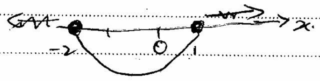
$= -8 + 24 - 24 + 3$

$= -5$

$(-2, -5)$

(b) $x^2 + x - 2 \le 0$

$(x+2)(x-1)$



$\therefore -2 \le x \le 1$

(c) $7\overline{)1000}$ → 142 r 6

$\therefore 994$ — T_{142}

$S_n = \frac{n}{2}(a + l)$

$= \frac{142}{2}(7 + 994)$

$= 71 \times 1001$

$= 71,071$

(d) $x^4 - 5x^2 + 4 = 0$

let $m = x^2$

$m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$\therefore m = 4$ or 1

$x^2 = 4$	$x^2 = 1$
$x = \pm 2$	$x = \pm 1$

(e) $y = x^2 + x$

$m_{tn} = \frac{dy}{dx} = 2x + 1$

$x = 1 \quad m = 3.$

Pt $(1, 2)$

tng: $y - y_1 = m(x - x_1)$

$y - 2 = 3x - 3$

$\therefore 3x - y + 1 = 0$

Norm: $y - 2 = -\frac{1}{3}(x - 1)$

$m = -\frac{1}{3}$

$3y - 6 = -x + 1$

$x + 3y - 7 = 0$

(f). $P = 50,000$

$R = 1.08$ $\quad A$ = amount left

$n = 20$

end of 1st yr $\quad A = 50,000 \times 1.08 - m$

$= PR - m.$

2nd yr $\quad A = (PR - m)R - m$

$= PR^2 - mR - m.$

3rd yr $\quad A = (PR^2 - mR - m)R - m$

$= PR^3 - m(R^2 + R + 1)$

20th yr $\quad 0 = PR^{20} - m(R^{19} + R^{18} + \dots + 1)$

$S_{20} = \frac{1(R^{20} - 1)}{R - 1}$

$0 = PR^{20} - m\left[\frac{R^{20} - 1}{R - 1}\right]$

$m = \frac{PR^{20}(R-1)}{R^{20} - 1} = \frac{50,000 \times 1.08^{20}(.08)}{1.08^{20} - 1}$

$= \$5,092.61$

(4) (f) (iv) $M = 8000$

We need R such that $50000R^{20} - 8000\left(\frac{R^{20}-1}{R-1}\right) = 0$

Test r = 10% ie $R = 1\cdot 1$

$$50000R^{20} - 8000\left(\frac{R^{20}-1}{R-1}\right) \approx -121\,825\cdot 00$$

Test r = 14% ie $R = 1\cdot 15$

$$50000R^{20} - 8000\left(\frac{R^{20}-1}{R-1}\right) \approx -41\,024\cdot 93$$

Test r = 15% ie $R = 1\cdot 15$

$$50000R^{20} - 8000\left(\frac{R^{20}-1}{R-1}\right) \approx -1221\cdot 79$$

Test r = 16% ie $R = 1\cdot 16$

$$50000R^{20} - 8000\left(\frac{R^{20}-1}{R-1}\right) \approx 50\,000\cdot 00$$

So 15% pa is the required interest rate.

NB As a point of interest a more exact answer is:

$$R = 1\cdot 150269696 \Rightarrow r = 15\cdot 0269696\%$$